*Properly establishing the relation between linear algebra and geometry makes it easier to obtain the three-by-three orthogonal matrix that describes a specified rotation.*

# A Note on Rotation Matrices

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The connection between three-by-three orthogonal matrices and the rotations of space that they describe is quite misleading when trying to describe a rotation of space by angles of rotation about the three coordinate axes. If the relation between the linear algebra and the geometry is properly established, not only do obscurities vanish but less computation is required to obtain the matrix describing a specified rotation.

This article will describe all three-by-three orthogonal matrices, what they do geometrically, and how to obtain directly a matrix having prescribed geometric properties. The arguments leading to the formulas are included for completeness, but the goals are the statements that connect these formulas with the geometry.

## Rotations

This discussion is restricted to rigid motions of space that fix at least one point: *rotations;* a fixed point is chosen to be the origin. Such a rigid motion is then given by a three-by-three matrix $A$: The row vector $x$ is sent to $xA$. Since angles are preserved, $(xA)(yA)^T$ always equals $xy^T$, where $T$ denotes transpose, so $xAA^Ty^T=xy^T$. That is, $AA^T=I$ is the identity; $A$ is an *orthogonal matrix.* Conversely, orthogonal matrices yield rotations.

The rows of an orthogonal matrix $A$ are $iA$, $jA$, $kA$, the images of the axis vectors $i = [1,0,0]$, etc., and these rows are orthonormal vectors, since $AA^T=I$. From $AA^T=I$ we have $A^T=A^{-1}$ and $A^TA=I$, so the columns of $A$ are orthonormal also.

The determinant of an orthogonal matrix $A$ is $+1$ or $-1$, since $AA^T=I$ gives $(\det A)^2=1$. The rotation is called *proper* or *improper* (or occasionally direct or opposite), as $\det A$ is $+1$ or $-1$. Proper rotations, usually called simply rotations, are the most useful. Let's discuss these first.

## Proper rotations—matrix known

The characteristic polynomial of a three-by-three matrix $A$ is

$$\det(\lambda I-A)=\lambda^3-(\mathrm{tr}\,A)\lambda^2+\ldots-\det A$$

where $\mathrm{tr}\,A$ is the trace of $A$ and the coefficient of $\lambda$ is not needed. When $A$ is proper orthogonal, $\det A = +1$, and this polynomial has at least one positive real root $\lambda>0$. If $\ell$ is an eigenvector accompanying $\lambda$, $\ell A=\lambda\ell$, and $(\ell A)(\ell A)^T=(\lambda\ell)(\lambda\ell)^T$ equals $\ell\ell^T$, so $\lambda^2=1$. Thus, $\lambda=+1$ and $\ell A=\ell$. $\ell$ is the axis of rotation of $A$; $A$ represents a rigid motion of space about a line through the origin in the direction of $\ell$.


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The angle $\theta$ of the rotation is the angle through which any vector in the plane through the origin and perpendicular to $\ell$ is moved. We can change the length of $\ell$ to length 1 and choose $e$ and $f$ orthogonal to $\ell$ so that $e, f, \ell$ is orthonormal and right-handed. Then $eA = (\cos\theta)e + (\sin\theta)f$ and similarly $fA$. If the rotation is referred to this basis, its matrix is

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which we denote by $R$. This means that $A = PRP^{-1}$, where $P$ is the change-of-basis matrix; $P$ expresses $e, f$, and $\ell$ in terms of $i, j, k$—that is, $e, f$, and $\ell$ are the rows of $P$. It is not necessary to know $P$ to determine the angle $\theta$; for $\operatorname{tr} A = \operatorname{tr}(PRP^{-1}) = \operatorname{tr} R = 1 + 2\cos\theta$.

The eigenvalues of a three-by-three matrix $A$ can be easily found by the following observations. If $B$ is any three-by-three matrix, let $B^a$ be its matrix of cofactors: The $ij$th entry is $(-1)^{i+j}$ times the two-by-two determinant obtained by striking the $j$th row and $i$th column from $B$. $B^a$ is commonly called the adjoint or adjugate of $B$. One has $B^aB = (\det B)I$. If $\det B \neq 0$, this gives Cramer's rule $B^{-1} = B^a/\det B$; and if $\det B = 0$, then $B^aB = 0$ shows any row $\ell \neq 0$ of $B^a$ satisfies $\ell B = 0$. Eigenvectors $\ell$ of $A$ for eigenvalue $\lambda$, $\ell A = \lambda\ell$ or $\ell(A - \lambda I) = 0$, are thus obtained as the rows of $(A - \lambda I)^a$. Especially, if $A$ is proper orthogonal, its axis is obtained as a row of $(A - I)^a$. If $\lambda$ is not a repeated eigenvalue of $A$, then $A - \lambda I$ has rank two and all rows of $(A - \lambda I)^a$ are in fact scalar multiples of one of them.

**Formulas:** A proper orthogonal matrix $A$ ($AA^T = I$ and $\det A = +1$) represents a rotation of space about a fixed axis. An eigenvector of $A$ for the eigenvalue 1 is in the direction of the axis and is obtained as any row of the matrix of cofactors $(A - I)^a$. The angle $\theta$ of rotation about this axis is given by $\operatorname{tr} A = 1 + 2\cos\theta$.

The desired eigenvector can be obtained even more simply as any row of the symmetric matrix $(A + A^T) - (\operatorname{tr} A - 1)I$. This will be shown later.

The sign of the angle $\theta$ changes when the eigenvector $\ell$ is replaced by $-\ell$, since this reverses the direction along the axis used for the usual right-hand rule. The proper choice of sign for $\theta$ is that of the determinant $\det(x, xA, \ell)$, where $x$ is any vector not parallel to $\ell$. Because, if we make $\ell$ of length 1, choose $e$ in the plane of $x$ and $\ell$, and choose $f$ so that $e, f, \ell$ is orthonormal and right-handed as before, then $x = ue + w\ell$, and from $eA = (\cos\theta)e + (\sin\theta)f$ we obtain $\det(x, xA, \ell) = u^2 \sin\theta$.

Example: One can use these formulas to show that the product of rotations through angles of 10 degrees, 20 degrees, and 30 degrees about the $x$-, $y$-, and $z$-axes respectively, done in that order, is the same as a single rotation through an angle of 35.8 degrees about an axis with direction cosines 0.124, 0.616, and 0.778. In general, this product of rotations about axes through angles $\alpha$, $\beta$, and $\gamma$ is a rotation through an angle $\theta$ given by $1 + 2\cos\theta = \cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta + \sin\alpha\sin\beta\sin\gamma$.

## Proper rotations—axis and angle known

A proper rotation of space is determined by its axis and angle. The problem, opposite that of the last section, is to find the proper orthogonal matrix having a prescribed vector determining the axis of the rotation and rotating by a prescribed angle about that axis. A formula for that matrix can be obtained by elementary methods (Rogers and Adams,[1] and Pavlidis[2]), but the argument to follow, which uses power series of matrices (Nobel and Daniel[3]), is chosen because it is simplest and the technique is useful in general.

Given the vector $\ell = [a, b, c]$ along the axis of rotation and of length $\lambda = \sqrt{a^2 + b^2 + c^2}$ not necessarily 1, introduce the matrix

$$L = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

This matrix represents the linear transformation of cross product with $\ell$; that is, $xL = -x \times \ell$. Indeed, the cross product of $x$ by $\ell$ is obtained by projecting $x$ onto the plane perpendicular to $\ell$ and rotating through a right angle there. The cross product rule $(x \times \ell) \times m = -x(\ell \cdot m) + (x \cdot m)\ell$ with $m = \ell$ gives $xL^2 = -x\lambda^2 + x\ell^T\ell$ for every $x$, so

$$L^2 + \lambda^2 I = \ell^T\ell = \begin{bmatrix} a \\ b \\ c \end{bmatrix} [a\ b\ c]$$

Aside: The same cross product rule also gives $x \times (\ell \times m) = (x \times \ell) \times m - (x \times m) \times \ell$, so that the matrix corresponding to $\ell \times m$ is the "bracket" $ML - LM$.

Note that $\ell L = 0$, so $L^3 + \lambda^2 L = (L^2 + \lambda^2 I)L = \ell^T\ell L = 0$, and we have

$$L^3 = -\lambda^2 L \qquad L^6 = \lambda^4 L^2$$
$$L^4 = -\lambda^2 L^2 \qquad L^7 = -\lambda^6 L$$
$$L^5 = \lambda^4 L \qquad \vdots$$

The exponential of a matrix is obtained by replacing $x$ with a matrix in the usual series $\exp x = 1 + x + x^2/2! + x^3/3! + \ldots$. Every partial sum is a matrix, and the sequences of entries converge. The argument now proceeds analogous to that of Euler's formula $e^{\sqrt{-1}\,t} = \cos t + \sqrt{-1}\sin t$ by power series of complex numbers.

Using the familiar series for cosine and sine, we have

$$\exp tL = I + tL + \frac{t^2}{2!}L^2 + \frac{t^3}{3!}L^3 + \frac{t^4}{4!}L^4 + \frac{t^5}{5!}L^5 + \frac{t^6}{6!}L^6 + \ldots$$

$$= I + tL + \frac{t^2}{2!}L^2 - \frac{t^3}{3!}\lambda^2 L - \frac{t^4}{4!}\lambda^2 L^2 + \frac{t^5}{5!}\lambda^4 L + \frac{t^6}{6!}\lambda^4 L^2 - \ldots$$

$$= I + \left(t - \lambda^2\frac{t^3}{3!} + \lambda^4\frac{t^5}{5!} - \ldots\right)L + \left(\frac{t^2}{2!} - \lambda^2\frac{t^4}{4!} + \lambda^4\frac{t^6}{6!} - \ldots\right)L^2$$



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or

$$\exp tL = I + \frac{\sin \lambda t}{\lambda} L + \frac{1 - \cos \lambda t}{\lambda^2} L^2$$

Aside: If $\ell = [0,0,1]$, then $L = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\lambda = 1$, so the formula gives

$$\exp t \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (\sin t) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ (1 - \cos t) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (\cos t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\sin t) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is Euler's formula with $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in the $xy$ plane playing the role of $\sqrt{-1}$.

We routinely verify that

- $\exp tL$ is an orthogonal matrix, since $L^T = -L$;
- $\det(\exp tL) = +1$, since $\operatorname{tr} L = 0$ and, for any matrix $A$, $\det(\exp A) = e^{\operatorname{tr} A}$ (see note below);
- $\exp tL$ has $\ell$ as an eigenvector with eigenvalue 1, since $\ell L = 0$; and
- $\operatorname{tr}(\exp tL) = 1 + 2\cos \lambda t$, since $\operatorname{tr} L^2 = -2\lambda^2$.

Thus, $\exp tL$ is the matrix of a proper rotation having axis $\ell$ and angle $\lambda t$.

**Formula:** The proper orthogonal matrix representing a rotation of space through an angle $\theta$ about an axis in the direction $[a, b, c]$ is given by

$$I + \frac{\sin \theta}{\lambda} L + \frac{1 - \cos \theta}{\lambda^2} L^2$$

where

$$L = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \text{ and } \lambda = \sqrt{a^2 + b^2 + c^2}$$

Note that this gives the desired matrix from the axis and angle given by $a$, $b$, $c$, and $\theta$ with very little computation.

This formula (in less modern dress) for the parametric specification of rotations about a point dates back to Euler.

A consequence of this formula is a simple way to obtain the eigenvector $\ell$ for eigenvalue 1 of the proper orthogonal matrix $A$. This is because $A = I + (\sin \theta/\lambda) L + ((1 - \cos \theta)/\lambda^2) L^2$ together with $\operatorname{tr} A = 1 + 2\cos \theta$, $L^2 + \lambda^2 I = \ell^T \ell$, and the fact that $L$ is skew-symmetric and $L^2$ is symmetric, gives

$$(A + A^T) - (\operatorname{tr} A - 1) I = \frac{3 - \operatorname{tr} A}{\lambda^2} \ell^T \ell$$

This symmetric matrix is of rank one, and its rows are scalar multiples of $\ell$. The skew-symmetric part of $A$ is a scalar multiple of $L$.

## Reflections

The transformation that reverses a vector $\ell$ and fixes every vector in the plane (through the origin) perpendicular to $\ell$ is the *reflection* in that plane. The projection of the (row) vector $x$ on $\ell$ has length $x\ell^T/\ell\ell^T$, so elementary geometry shows that the reflection sends $x$ to $x - 2(x\ell^T/\ell\ell^T)\ell = x(I - 2(\ell^T\ell/\ell\ell^T))$ and thus is linear and described by the matrix $A = I - 2(\ell^T\ell/\ell\ell^T)$. Using $L^2 + \lambda^2 I = \ell^T\ell$ of the previous section, we can even write $A = -(I + 2(L^2/\lambda^2))$.

The matrix $A$ of a reflection satisfies $A^2 = I$ and $A^T = A$; so $A^T = A^{-1}$, and $A$ is orthogonal. With respect to a basis consisting of two perpendicular vectors perpendicular to $\ell$ and $\ell$ itself, the reflection has matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and hence $\det A = -1$. A reflection is an improper rotation.

The central theoretical role of reflections will be discussed later.

## Improper rotations

When the three-by-three matrix $A$ is improper orthogonal, $\det A = -1$ and the characteristic polynomial

$$\det(\lambda I - A) = \lambda^3 - (\operatorname{tr} A)\lambda^2 + \ldots - \det A$$

has at least one negative real root $\lambda < 0$. As before, $\ell A = \lambda\ell$ gives $\lambda^2 = 1$, so $\lambda = -1$; $A$ has an eigenvector $\ell$, which it reverses.

Let $B$ be the reflection in the plane perpendicular to this $\ell$. Then $AB$ fixes $\ell$ and is a proper rotation through an angle $\theta$. Using two vectors perpendicular to $\ell$ and $\ell$ as basis, $(AB)B = A$ becomes

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus $A$ is a *rotatory reflection*—a rotation about an axis together with a simultaneous reflection in the plane perpendicular to that axis. The angle and axis are obtained by trace and eigenvector of $A$ as before.

**Formulas:** An improper orthogonal matrix $A$ ($AA^T = I$ and $\det A = -1$) represents simultaneously a rotation about an axis and a reflection in the plane perpendicular to that axis. An eigenvector of $A$ for the eigenvalue $-1$ is in the direction of the axis and is obtained as any row of the matrix of cofactors $(A + I)^a$. The angle $\theta$ of the rotation is obtained by $\operatorname{tr} A = -1 + 2\cos\theta$.

This last formula is obvious when the matrix is diagonal and easy if it is upper triangular. In general, $A = PJP^{-1}$ with $J$ upper triangular and $P$ and $J$ possibly complex. Then $\exp A = P(\exp J)P^{-1}$, so $\det(\exp A) = \det(P(\exp J)P^{-1}) = \det(\exp J) = e^{\operatorname{tr} J} = e^{\operatorname{tr} A}$.



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Changing a rotation matrix $A$ to $-A$ interchanges proper and improper, changes the sign of the rotation angle, but does not change the axis. From this we obtain

$$-I + \frac{\sin\theta}{\lambda} L - \frac{1+\cos\theta}{\lambda^2} L^2, \; L = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

which is the matrix of a rotatory reflection about an axis in the direction $[a, b, c]$ through the angle $\theta$.

## Products of reflections

The product of the two reflections in planes perpendicular to $\ell$ and $m$ is a proper rotation with axis in the direction perpendicular to both $\ell$ and $m$ and through an angle twice that between $\ell$ and $m$. This is an easy geometric fact. It can also be argued using matrices by writing the matrices for the reflections as $A = -(I + 2(L^2/\lambda^2))$ and $B = -(I + 2(M^2/\mu^2))$, recalling that the angle of rotation is found by $1 + 2\cos\theta = \mathrm{tr}(AB)$, and computing $\mathrm{tr}(L^2M^2)$ from $(L^2 + \lambda^2 I)(M^2 + \mu^2 I) = \ell^T \ell m^T m$. Conversely, every proper rotation is the product of two reflections; we simply choose $\ell$ and $m$ perpendicular to the axis and separated by a suitable angle.

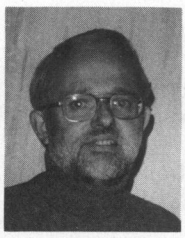
From the argument in the last section, we know that an improper rotation is the product of a reflection and a proper rotation, so three reflections suffice to obtain improper rotations.

**Theorem:** Every rotation of space (proper or improper) is the product of no more than three reflections.

This is a special case of a theorem going back to E. Cartan. A general formulation can be found in the beautiful but difficult book by Artin.[4] An elementary discussion of direct and opposite isometries of space that includes translations and their kin appears in Coxeter.[5] The utility of this particular theorem for computer graphics is not the statement but the connection with linear algebra, as discussed here. ■

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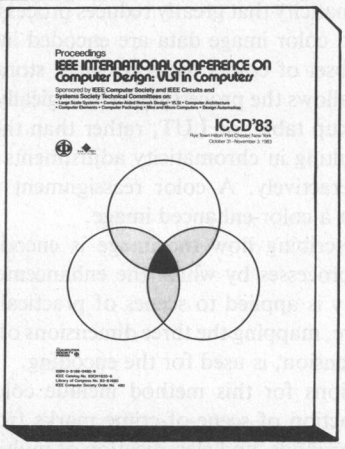
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